Topic 1

What is a differential equation?

Def:

- An equation relating an unknown function and one or more of its derivatives is called a <u>differential equation</u> (DE).
- If the unknown function depends on only a single independent variable then the equation is called an <u>ordinary differential equation</u> (ODE)
- If the unknown function depends on two or more variables and contains partial derivatives then the equation is called a <u>partial differential equation</u> (PDE)
- The <u>order</u> of a differential equation is the order of the highest derivative that appears in it.

Ex:

$$\boxed{\frac{dy}{dx} = y^2}$$

ODE

$y = y(x)$ is a function of one variable x

$y \leftarrow$ dependent variable

$x \leftarrow$ independent variable

order is 1

Ex:

$$\boxed{4x^2 y'' + y = 0}$$

ODE

$y = y(x)$ is a function of one variable x

$y \leftarrow$ dependent variable

$x \leftarrow$ independent variable

order is 2

Ex:

$$\boxed{\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}}$$

PDE

$u = u(x,t)$ is a function of two variables x and t

$u \leftarrow$ dependent variable

$x, t \leftarrow$ independent variables

order is 2

Def: An ODE is called linear if it is of the form

$$\underbrace{a_n(x)}\, y^{(n)} + \underbrace{a_{n-1}(x)}\, y^{(n-1)} + \dots + \underbrace{a_1(x)}\, y' + \underbrace{a_0(x)}\, y = \underbrace{b(x)}$$

That is, the coefficient's of the $y^{(k)}$ terms only have x's and constants in them

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Ex:

$$\underbrace{x^2}\, y'' - \underbrace{2}\, y' + \underbrace{\ln(x)}\, y = \underbrace{5x}$$

only has x's and constants

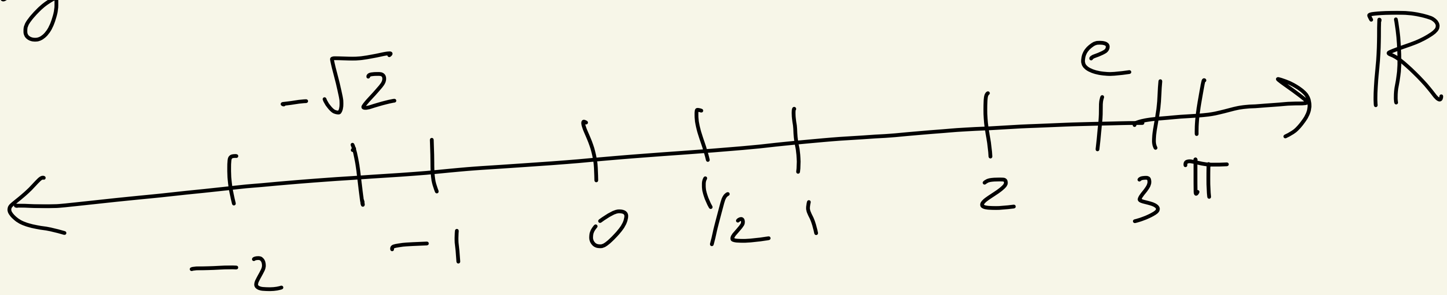
linear

$$\underbrace{y^2}\, y'' - 5y' + 12xy = 0$$

has y^2 as coefficient

non-linear

Def: The real number line is denoted by $\mathbb{R}$.



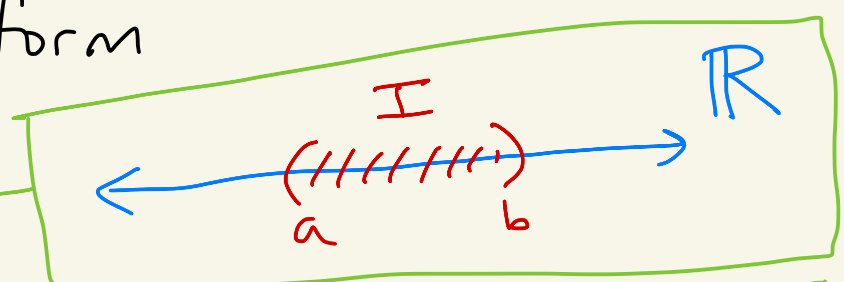
Def: An <u>open interval</u> I is an interval of the form

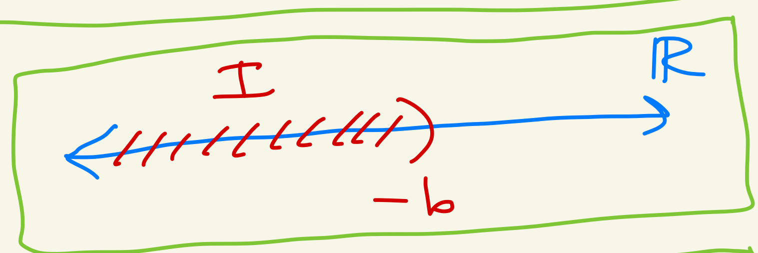
$I = (a, b)$

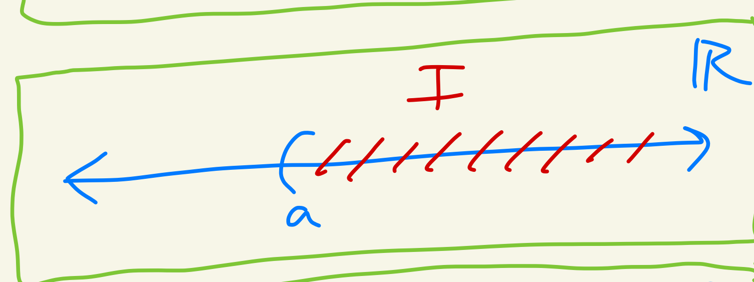
or $I = (-\infty, b)$

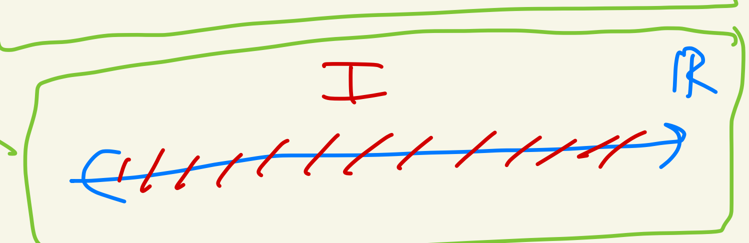
or $I = (a, \infty)$

or $I = \mathbb{R} = (-\infty, \infty)$









Def: A real-valued function f is a solution to an n-th order ODE on an open interval I if

1. $f, f', f'', \ldots, f^{(n)}$ exist on I

and 2. When you plug f into the ODE it solves the equation for all x in I.

In addition, one sometimes is given what $f(x_0), f''(x_0), \ldots, f^{(n-1)}(x_0)$ must equal for some fixed x_0 in I. In that case the problem is called an initial value problem.

Ex: Show that $f(x) = e^x$ is a solution to the ODE

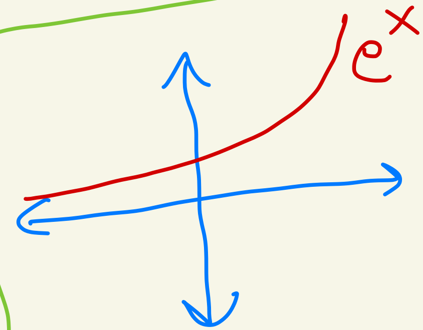
$$y' - y = 0$$

linear ODE
order is 1

on the interval $I = (-\infty, \infty)$.

① $f(x) = e^x$
$f'(x) = e^x$

both are defined for all x, ie on $I = (-\infty, \infty)$



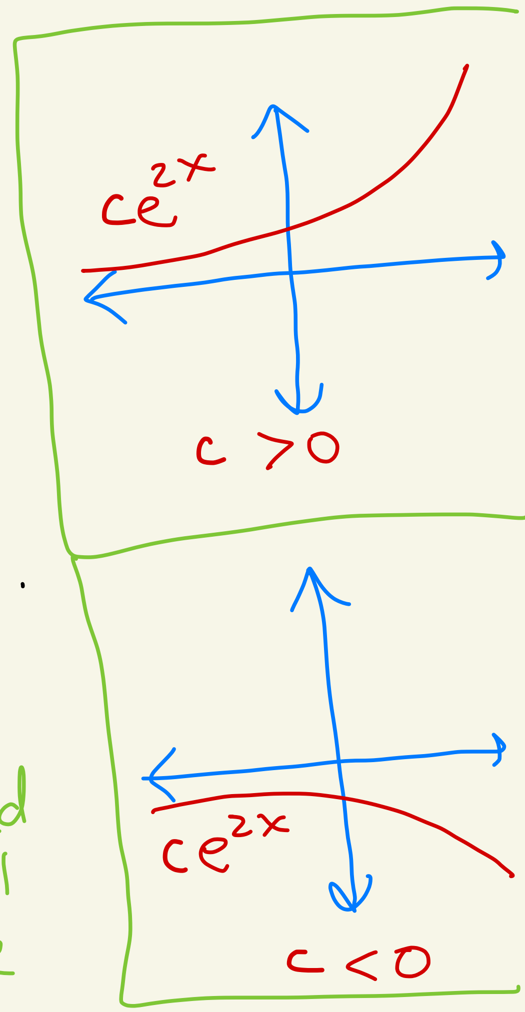
② We must show that $f'(x) - f(x) = 0$ is true for all x in I.
This is true since $e^x - e^x = 0$ for all x in I.

$\blacksquare$

Ex: Consider the ODE

$$y'' - 4y = 0$$

Let's try to find a solution on $I = (-\infty, \infty)$.



Let $f(x) = ce^{2x}$ where c is a constant.

Then,

$$\left.\begin{aligned} f(x) &= ce^{2x} \\ f'(x) &= 2ce^{2x} \\ f''(x) &= 4ce^{2x} \end{aligned}\right\}$$

defined for all x, ie on $I = (-\infty, \infty)$

We must show that

$$f''(x) - 4f(x) = 0$$

for all x in $I = (-\infty, \infty)$

This is true since $4ce^{2x} - 4ce^{2x} = 0$ for all x in I.

Thus, we have found an infinite number of solutions to $y'' - 4y = 0$, one for each constant c.

Some examples are $5e^{2x}, 10e^{2x}, -\pi e^{2x}, \ldots$

What if we were asked to find a solution to the initial value problem:

$$\boxed{\begin{aligned} & y'' - 4y = 0 \\ & y(0) = 5 \\ & y'(0) = 10 \end{aligned}}$$

← same ODE

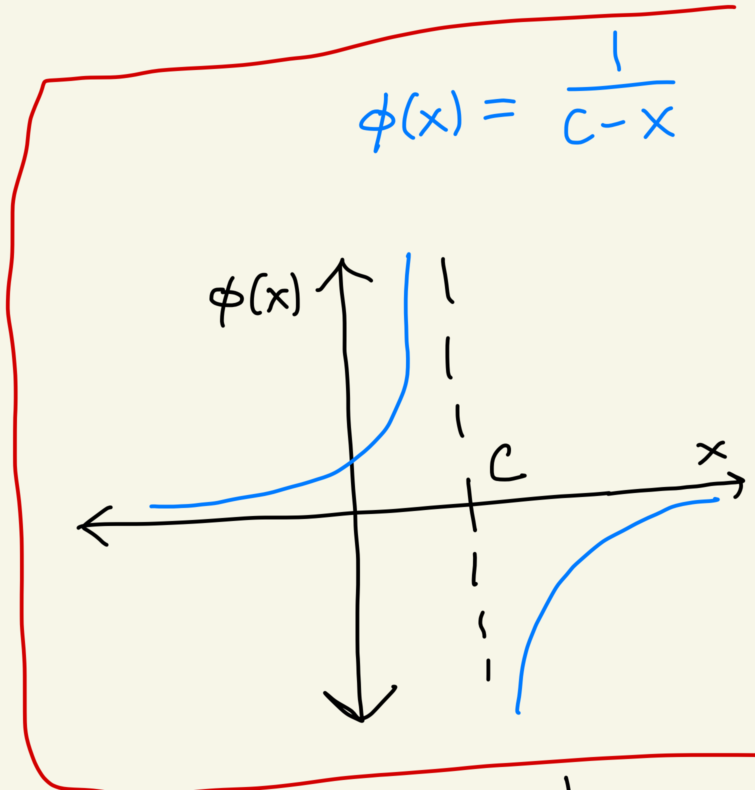
initial conditions

A solution to this problem is $f(x) = 5e^{2x}$ since $f(0) = 5e^{2(0)} = 5$

$f'(0) = 10e^{2(0)} = 10$

Ex: Consider the non-linear first order ODE

$$\boxed{\frac{dy}{dx} = y^2} \quad (*)$$



Let $\phi(x) = \frac{1}{C-x}$

where C is a constant.

Then,

$$\phi'(x) = \left((C-x)^{-1}\right)' = -(C-x)^{-2} \cdot (-1) = \frac{1}{(C-x)^2}$$

$$(\phi(x))^2 = \left(\frac{1}{C-x}\right)^2 = \frac{1}{(C-x)^2}$$

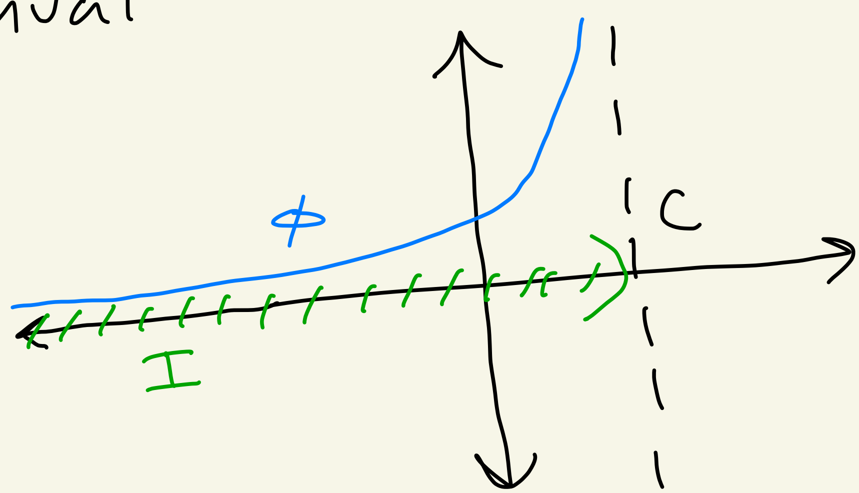
Thus, plugging ϕ in for y and ϕ' in for $\frac{dy}{dx}$ in (**) gives

$$\frac{1}{(C-x)^2} = \frac{1}{(C-x)^2}$$

That is, $\phi(x) = \frac{1}{C-x}$ satisfies $(*)$.

Note that $\phi(x)$ and $\phi'(x)$ only exist for $x \neq c$.

Thus, for example, $\phi(x) = \frac{1}{c-x}$ is a solution to $(*)$ on the interval $I = (-\infty, c)$.



You could also say that ϕ is a solution to $(*)$ on $I = (c, \infty)$.

Ex: Solve

$$\left.\begin{aligned} &\frac{dy}{dx} = y^2 \\ &\text{where } y(0) = 1. \end{aligned}\right\} \quad (**)$$

We know that $\phi(x) = \dfrac{1}{c-x}$ solves $\dfrac{dy}{dx} = y^2$ for $x \neq c$.

Let's try to solve $\phi(0) = 1$.

We get $\dfrac{1}{c-0} = 1$ which gives $c = 1$.

Thus, $\phi(x) = \dfrac{1}{1-x}$ solves $(**)$ on the interval $I = (-\infty, 1)$